South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.14, No.1 2018, pp. 103-114

ON INNER PRODUCT SPACES OVER ZEROSUMFREE SEMIRINGS

Sushobhan Maity

Department of Mathematics,
Visva-Bharati, Santiniketan-731235, INDIA
E-mail: susbhnmaity@gmail.com

(Received: November 03, 2017)

Abstract: A semiring A is called a zerosumfree semiring if for all $a, b \in A$, $a+b = 0$ implies that $a = 0$ and $b = 0$. Here we have endowed every semimodule over a zerosumfree semiring A with a natural inner product of values in A. A dimension theorem for orthonormal bases of these inner product spaces over zerosumfree semiring is proved. The main results in this paper generalize the corresponding results on the Boolean inner product spaces [5].

Keywords and Phrases: Semiring, zerosumfree, inner product, orthonormal, isometry.

2010 Mathematics Subject Classification: 16Y60, 15A03, 15A04.

1. Introduction and Preliminaries

Linear algebra over distributive lattices, Boolean algebras, incline algebras and semirings etc. has a long history, as well as holds an important position in the modern theory of linear algebra due to their applications to computer science, optimizations and theoretical physics [3,7]. Several authors have studied invertibility of matrices and linear operators on the vector spaces over such algebras [2,6,9,10,11,12,13,15], whereas some others are developing the theory of vector spaces over such algebras [14,16,17]. In 2009, Gudder and Latremoliere [5] developed inner product spaces over a Boolean algebra. Gudder and Latermoliere [5] generalized some results on power of matrices over the two element Boolean algebra $\{0, 1\}$ [4] to arbitrary Boolean algebras. In this paper following the idea of Gudder and Latermoliere, we have developed inner product spaces on zerosumfree semirings. Since every Boolean algebra is a zerosumfree semiring, our results generalize the corresponding results on the Boolean inner product spaces. It is a worthy generalization, since it enables us to get the results without two strong properties of Boolean algebras like absorptive property and having complement of each element.

A semiring is an algebraic system $(S, +, \cdot)$ such that both $(S, +)$ and $(S, \cdot)$ are commutative monoids with identity elements 0 and 1 respectively, and connected by the ring-like distributive laws. Also, the zero element 0 is absorbing in S, i.e., $0r = r0 = 0$ for all $r \in S$. Let S be a semiring and $a \in S$. We denote by a^k the k-th power of a and by ka the sum $a + a + \cdots + a$(k times) for any positive integer k. A semiring S is called a zerosumfree semiring if $a + b = 0$ implies that $a = b = 0$ for any $a, b \in S$. Zerosumfree semirings are also known as antirings [3]. They are quite abundant: for example, every Boolean algebra, the fuzzy algebra $([0, 1], \vee, T)$, where T is a $t-$ norm [8], every distributive lattice and every incline [1] are zerosumfree semirings. The set $\mathbb{Z}^+$ of all nonnegative integers with the usual operations of addition and multiplication of integers is a zerosumfree semiring. The max-plus algebra $(\mathbb{R} \cup \{-\infty\}, max, +)$ and min-plus algebra $(\mathbb{R} \cup \{+\infty\}, min, +)$ are commutative zerosumfree semirings. For more details on zerosumfree semirings we refer to [3,7,15].

Here we initiate to develop the semimodules over zerosumfree semirings. We have introduced the notion of orthogonality and proved that the definition of inner product in a semimodule over a zerosumfree semiring, is independent of the choice of basis. At last we have shown that any two bases of a semimodule over a zerosumfree semiring have the same cardinality.

Before going to the abstract inner product spaces over a zerosumfree semiring in general, we first characterize the particular inner product space $\mathcal{L}_n(A)$ of all n-tuples on a zerosumfree semiring A.

2. The Inner Product Space $\mathcal{L}_n(A)$

In this section we characterize $\mathcal{L}_n(A)$, the set of all $n-$tuples of elements in A and which is endowed with an inner product. Here we show that any orthonormal basis of $\mathcal{L}_n(A)$ has exactly n elements.

Let A be a zerosumfree semiring. For all $n \in \mathbb{N}$, we denote by $\mathcal{L}_n(A)$ the set of all n-tuples of elements in A. We endow $\mathcal{L}_n(A)$ with the following operations: if $\underline{a}=(a_1, a_2, \cdots, a_n)$, $\underline{b}=(b_1, b_2, \cdots, b_n) \in \mathcal{L}_n(A)$ and $c \in A$, then

$$\begin{aligned} \underline{a} + \underline{b} &= (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n) \\ \text{and} \quad c\underline{a} &= (ca_1, ca_2, \cdots, ca_n). \end{aligned}$$

Then $\mathcal{L}_n(A)$ satisfies all axioms of a linear space regarding the interrelations be-

tween the external composition $A \times \mathcal{L}_n(A) \longrightarrow \mathcal{L}_n(A)$ and the semigroup structure of $(\mathcal{L}_n(A), +)$ and the semiring structure of $(A, +, \cdot)$.
We call the elements of $\mathcal{L}_n(A)$ zerosumfree vectors and call $\mathcal{L}_n(A)$ a zerosumfree semimodule. We will use the following definitions throughout the rest of this chapter.

The zerosumfree semimodule $\mathcal{L}_n(A)$ is endowed with a natural inner product.

Definition 2.1. *Let* $\underline{a}=(a_1, \cdots, a_n)$ *and* $\underline{b}=(b_1, \cdots, b_n)$ *be two vectors of* $\mathcal{L}_n(A)$. *Then we define the inner product of these two vectors by*

$$\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^{n} a_i b_i.$$

Definition 2.2. *Two vectors* $\underline{a}$ *and* $\underline{b}$ *in* $\mathcal{L}_n(A)$ *are orthogonal when* $\langle \underline{a}, \underline{b} \rangle = 0$, *in which case we shall write* $\underline{a} \perp \underline{b}$.

The vector $\underline{a}$ *is said to be a unit vector when* $\langle \underline{a}, \underline{a} \rangle = 1$.

Definition 2.3. *An orthogonal set in* $\mathcal{L}_n(A)$ *is a subset* E *of* $\mathcal{L}_n(A)$ *such that for all* $\underline{e}, \underline{f} \in E$ *we have* $\underline{e} \neq \underline{f} \Rightarrow \langle \underline{e}, \underline{f} \rangle = 0$. *An orthonormal subset of* $\mathcal{L}_n(A)$ *is an orthogonal set whose all elements are unit vectors.*

Definition 2.4. *A subset* B *of* $\mathcal{L}_n(A)$ *is a generating subset of* $\mathcal{L}_n(A)$ *when every element of* $\mathcal{L}_n(A)$ *is a linear combination of elements in* B.
A subset B *of* $\mathcal{L}_n(A)$ *is called a basis of* $\mathcal{L}_n(A)$ *when every element of* $\mathcal{L}_n(A)$ *can be expressed uniquely as a linear combination of elements in* B *with nonzero coefficients. If moreover* B *is orthonormal, then it is called an orthonormal basis of* $\mathcal{L}_n(A)$.

Proposition 2.5. *If* $\underline{a} = (a_1, \cdots, a_n)$ *and* $\underline{b} = (b_1, \cdots, b_n)$ *in* $\mathcal{L}_n(A)$ *are orthogonal then* $a_i b_i = 0$ *for all* $i = 1, 2, \cdots, n$.
Proof. In $\mathcal{L}_n(A)$, $\underline{a} \perp \underline{b}$ implies that $a_1b_1 + a_2b_2 + \cdots + a_nb_n = 0$. Since A is zerosumfree, $a_1b_1 = a_2b_2 = \cdots = a_nb_n = 0$.

Lemma 2.6. *Let* M *be an orthonormal set in* $\mathcal{L}_n(A)$. *Then every linear combination of elements of* M *has unique expression.*
Proof. Let $\underline{a}(\neq \underline{0}) \in \mathcal{L}_n(A)$ be such that $\underline{a} = \sum_{i=1}^{m} c_i \underline{e_i} = \sum_{i=1}^{n} d_i \underline{f_i}$, where $c_1, c_2, \cdots, c_m; d_1, d_2, \cdots, d_n \in A \setminus \{0\}$ and $\underline{e_1}, \underline{e_2}, \cdots, \underline{e_m}; \underline{f_1}, \underline{f_2}, \cdots, \underline{f_n} \in M$. Since M is orthonormal, $c_i = \langle \sum_{i=1}^{m} c_i \underline{e_i}, \underline{e_i} \rangle$ for $i = 1, 2, \cdots, m$. If $\underline{e_j} \notin \{\underline{f_1}, \underline{f_2}, \cdots, \underline{f_n}\}$ for some $j \in \{1, 2, \cdots, m\}$, then $c_j = \langle \underline{a}, \underline{e_j} \rangle = \langle \sum_{i=1}^{n} d_i \underline{f_i}, \underline{e_j} \rangle = 0$, which contradicts our assumption that $c_i \neq 0$ for all $i \in \{1, 2, \cdots, m\}$. Consequently, $\{\underline{e_1}, \underline{e_2}, \cdots, \underline{e_m}\} \subseteq \{\underline{f_1}, \underline{f_2}, \cdots, \underline{f_n}\}$. The reverse inclusion follows similarly. Thus

$m = n$ and upon renumbering we have $\underline{e_i} = \underline{f_i}$ for all $i = 1, 2, \cdots, n$. Then $c_i = \langle \underline{a}, \underline{e_i} \rangle = \langle \underline{a}, \underline{f_i} \rangle = d_i$. The case $\underline{a} = \underline{0}$ follows trivially. Hence the result follows.

Theorem 2.7. *If $M = \{\underline{a_1}, ..., \underline{a_m}\}$ is an orthonormal subset of $\mathcal{L}_n(A)$, then $m \leq n$.*

Proof. We prove it by induction on n, the numbers of tuples of elements of A. First, we prove it for n=1, suppose on the contrary, $\{a, b\}$ is an orthonormal subset of $\mathcal{L}_1(A)$. Then $a \cdot b = 0$ and $a^2 = 1 = b^2$. On multiplication by a on both sides of $a \cdot b = 0$, we get $b = 0$, similarly $a = 0$. This contradicts our assumption. Hence it is true for $n = 1$. Suppose the result is true for all $\mathcal{L}_k(A)$ where $k \leq n$.

Let $M = \{\underline{a_1}, \cdots, \underline{a_m}\}$ is an orthonormal subset of $\mathcal{L}_{n+1}(A)$ and $\underline{a_j} = (a_{1j}, a_{2j}, \cdots, a_{n+1,j})$. Consider the set $M' = \{\underline{c_2}, \cdots, \underline{c_m}\}$ defined by $\underline{c_j} = (c_{1j}, c_{2j}, ..., c_{nj})$ where $c_{ij} = a_{i1}a_{n+1,j} + a_{ij}$. Now we show that M' forms an orthonormal subset of $\mathcal{L}_n(A)$. First we show that for all $r \neq s$, $c_{1r}c_{1s} + c_{2r}c_{2s} + \cdots + c_{nr}c_{ns} = 0$. Now, since $r, s \neq 1$, we have

$$\begin{aligned} c_{ir}c_{is} =& (a_{i1}a_{n+1,r} + a_{ir})(a_{i1}a_{n+1,s} + a_{is}) \\ =& a_{i1}^2 a_{n+1,r}a_{n+1,s} + a_{i1}a_{n+1,r}a_{is} + a_{ir}a_{i1}a_{n+1,s} + a_{ir}a_{is} \\ =& 0 \qquad\qquad \text{by proposition (2.4)} \end{aligned}$$

and

$$\begin{aligned} \langle \underline{c_r}, \underline{c_r} \rangle =& c_{1r}^2 + c_{2r}^2 + \cdots + c_{nr}^2 \\ =& (a_{11}a_{n+1,r} + a_{1r})^2 + (a_{21}a_{n+1,r} + a_{2r})^2 + \cdots + (a_{n1}a_{n+1,r} + a_{nr})^2 \\ =& a_{n+1,r}^2(a_{11}^2 + a_{21}^2 + \cdots + a_{n1}^2) + 2a_{n+1,r}(a_{11}a_{1r} + a_{21}a_{2r} + \cdots + a_{n1}a_{nr}) \\ & \qquad + (a_{1r}^2 + a_{2r}^2 + \cdots + a_{nr}^2) \\ =& a_{n+1,r}^2(a_{11}^2 + a_{21}^2 + \cdots + a_{n1}^2 + a_{n+1,1}^2) + (a_{1r}^2 + a_{2r}^2 + \cdots + a_{nr}^2) \\ & \qquad (for\ r \neq 1,\ a_{n+1,r}a_{n+1,1} = 0) \\ =& a_{1r}^2 + a_{2r}^2 + \cdots + a_{nr}^2 + a_{n+1,r}^2 \\ =& 1 \end{aligned}$$

So M' is an orthonormal subset of $\mathcal{L}_n(A)$. Hence by induction hypothesis $m-1 \leq n$ i.e. $m \leq n+1$ and the result follows.

Theorem 2.8. *Let $M = \{\underline{a_1}, ..., \underline{a_n}\}$ be an orthonormal subset of $\mathcal{L}_m(A)$. Then M is an orthonormal basis for $\mathcal{L}_m(A)$ if and only if the rows of $(a_{ij})_{m\times n}$ is an orthonormal subset of $\mathcal{L}_n(A)$.*

Proof. Denote $\underline{a_j} = (a_{1j}, ..., a_{mj})$, $j = 1, ..., n$. Then we arrange M as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Assume that M is an orthonormal basis for $\mathcal{L}_m(A)$. Since $\underline{\delta_1} = (1, ..., 0) \in \mathcal{L}_m(A)$, there exist $b_1, b_2, \cdots, b_n \in A$ such that $\underline{\delta_1} = \sum_{j=1}^{n} b_j \underline{a_j}$, which implies

$$b_1 a_{11} + b_2 a_{12} + \cdots + b_n a_{1n} = 1 \tag{2.1}$$
$$b_1 a_{21} + b_2 a_{22} + \cdots + b_n a_{2n} = 0 \tag{2.2}$$
$$\vdots \tag{2.3}$$
$$b_1 a_{m1} + b_2 a_{m2} + \cdots + b_n a_{mn} = 0 \tag{2.4}$$

Since A is orthonormal, multiplying (2.1) by a_{11}, (2.2) by a_{21}, $\cdots$, (2.4) by a_{m1} and then adding we get $b_1 = a_{11}$. Similarly multiplying (2.1) by a_{12}, (2.2) by a_{22}, $\cdots$, (2.4) by a_{m2} and then adding we get $b_2 = a_{12}$. Similarly we get $b_3 = a_{13}, \cdots, b_n = a_{1n}$.
Putting these values of b_i's in (2.1), we get

$$\begin{aligned} (1, 0, \cdots, 0) &= a_{11}\underline{a_1} + a_{12}\underline{a_2} + \cdots + a_{1n}\underline{a_n} \\ &= a_{11}(a_{11}, \cdots, a_{m1}) + a_{12}(a_{12}, \cdots, a_{m2}) + \cdots + a_{1n}(a_{1n}, \cdots, a_{mn}) \end{aligned}$$

Equating componentwise, we get

$$\begin{aligned} & a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2 = 1 \\ \textit{and} \quad & \sum_{j=1}^{n} a_{1j} a_{ij} = 0, \quad (i \neq 1). \end{aligned}$$

Similarly taking $\underline{\delta_2}, \cdots, \underline{\delta_m}$, we see that

$$\begin{aligned} & a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2 = 1, \qquad (i = 2, \cdots, m). \\ \textit{and} \quad & \sum_{j=1}^{n} a_{ij} a_{rj} = 0, \qquad (i \neq r). \end{aligned}$$

Conversely, suppose that the rows of $(a_{ij})_{m \times n}$ is an orthonormal subset of $\mathcal{L}_n(A)$. That means $\sum_{j=1}^{n} a_{ij}^2 = 1$ for all $i = 1, \cdots, m$ and $\sum_{j=1}^{n} a_{kj} a_{ij} = 0$ for $k \neq i$;

$k, i \in \{1, 2, \cdots, m\}$.
Thus it follows that

$$\sum_{j=1}^{n} a_{kj}a_{ij} = \delta_{ki} = 1, \quad k = i$$
$$= 0, \quad k \neq i; \quad (k, i = 1, 2, \cdots, m).$$

It is equivalent to
$a_{k1}\underline{a_1} + a_{k2}\underline{a_2} + \cdots + a_{kn}\underline{a_n} = \underline{\delta_k}$, that is $\sum_{j=1}^{n} a_{kj}\underline{a_j} = \underline{\delta_k}$. Since $\{\underline{\delta_1}, \cdots, \underline{\delta_m}\}$ is the standard basis for $\mathcal{L}_m(A)$. So every vector in $\mathcal{L}_m(A)$ has unique expression and $\{\underline{a_1}, \cdots, \underline{a_n}\}$ generates $\mathcal{L}_m(A)$. It is also given that it is an orthonormal subset of $\mathcal{L}_m(A)$, consequently an orthonormal basis of $\mathcal{L}_m(A)$.

Theorem 2.9. *Let A be a zerosumfree semiring. Then any orthonormal basis of $\mathcal{L}_n(A)$ has exactly n elements.*
Proof. Let $M = \{\underline{a_1}, \underline{a_2}, \cdots, \underline{a_m}\}$ be an orthonormal basis of $\mathcal{L}_n(A)$. Then considering M as $(a_{ij})_{n\times m}$ matrix, by Theorem 2.7, we see that the rows of $(a_{ij})_{n\times m}$ is an orthonormal subset of $\mathcal{L}_m(A)$ and hence $n \leq m$ by Theorem 2.6.

Also since M is an orthonormal subset of $\mathcal{L}_n(A)$, then by Theorem 2.6, $m \leq n$. Thus $m = n$. Hence any orthonormal basis of $\mathcal{L}_n(A)$ has exactly n elements.

3. Inner product spaces on zerosumfree semiring

In this section we introduce abstract inner product space on a zerosumfree semiring. Given a zerosumfree semiring A and a semigroup $(V, +)$ we define V to be an inner product space over A as follows:

Definition 3.1. *A zerosumfree semimodule is a system $(V, A, +, \cdot)$, where $(V, +)$ is a commutative monoid with identity 0_v, A is a zerosumfree semiring and $'\cdot'$ is a map $A \times V \to V$ such that*

(i) $a \cdot (b \cdot v) = (ab) \cdot v$ *for all* $a, b \in A$ *and* $v \in V$,

(ii) $a \cdot (u + v) = a \cdot u + a \cdot v$ *for all* $a \in A$ *and* $u, v \in V$,

(iii) $(a + b) \cdot v = a \cdot v + b \cdot v$ *for all* $a, b \in A$ *and* $v \in V$,

(iv) $1.v = v$ *for all* $v \in V$,

(v) $0.v = a.0_v = 0_v$ *for all* $a \in A$ *and* $v \in V$,

(vi) *There exists a set of vectors* $\{v_1, \cdots, v_n\} \subseteq V$ *such that every* $v \in V$ *can be expressed as* $v = \sum a_i v_i$ *uniquely in the following sense: if for* $a_i, b_i \in A$

$$v = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i$$

then $a_i = b_i$, *for* $i = 1, \cdots, n$. *The set* $\{v_1, v_2, \cdots, v_n\}$ *of vectors is called a basis of* V.

$\mathcal{L}_n(A)$ is an example of a zerosumfree semimodule with a basis $\{(1, 0, \cdots, 0), (0, 1, 0 \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$. Which we call the standard basis of $\mathcal{L}_n(A)$.

Let V be a zerosumfree semimodule over A with basis $\{v_1, \cdots, v_n\}$. We define an inner product $\langle , \rangle : V \times V \longrightarrow A$ by: for $u = \sum a_i v_i$ and $v = \sum b_i v_i$,

$$\langle u, v \rangle = \sum a_i b_i. \tag{3.1}$$

Definition 3.2. *A zerosumfree semimodule* V *together with the above inner product is called a zerosumfree inner product space.*
The inner product on $\mathcal{L}_n(A)$,

$$\langle (a_1, a_2, \cdots, a_n), (b_1, b_2, \cdots, b_n) \rangle = \sum a_i b_i$$

as defined in section 2 coincides with the inner product as defined in Definition 3.1 if the basis considered is the standard basis. Thus $\mathcal{L}_n(A)$ is a zerosumfree inner product space.

Now we have following straightforward set of results:

Lemma 3.3. *Let* V *be an inner product space on a zerosumfree semiring* A. *Then for all* $u, v, w \in V$, $a \in A$ *we have:*

(i) $\langle u, v \rangle = \langle v, u \rangle$,

(ii) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

(iii) $\langle au, v \rangle = a\langle u, v \rangle$,

(iv) $\langle u, v \rangle = \langle w, v \rangle$ *for all* $v \in V$ *implies that* $u = w$,

Proof. We left the proof of $(i), (ii)$, and (iii).
(iv) Given that $\langle u, v \rangle = \langle w, v \rangle$ for all $v \in V$. Let $\{v_1, \cdots, v_n\}$ be a basis of V. Suppose that $u = \sum_{i=1}^{n} a_i v_i, v = \sum_{i=1}^{n} b_i v_i, w = \sum_{i=1}^{n} c_i v_i$.

Then from the condition we have $a_1b_1 + \cdots + a_nb_n = c_1b_1 + \cdots + c_nb_n$. It holds for any $v \in V$, so taking $v = v_1$ we get $a_1 = c_1$. Similarly taking $v = v_2$ we get $a_2 = c_2$. Proceeding similarly we see that $a_i = c_i$ for all $i = 1, \cdots, n$. Hence $u = w$.

Thus, the inner product is symmetric (i), linear ((ii) and (iii)), and nondegenerate (iv) .
Let V and W be two zerosumfree semimodule over a zerosumfree semiring A. Then a map $T : V \to W$ is called linear if $T(av) = aTv$ and $T(u + v) = Tu + Tv$ for all $u, v \in V$ and $a \in A$. A linear map T is said to be isometry if $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all $u, v \in V$.

If $T : V \longrightarrow W$ is an isometry then for $v_1, v_2 \in V$, $Tv_1 = Tv_2$ implies that $\langle v_1, v\rangle = \langle Tv_1, Tv\rangle = \langle Tv_2, Tv\rangle = \langle v_2, v\rangle$ for all $v \in V$. Hence by Lemma 3.3(iv), $v_1 = v_2$. Thus every isometry is injective.

By an isomorphism $T : V \longrightarrow W$ we mean a surjective isometry and then V and W are called isomorphic.

If $T : V \longrightarrow W$ is an isomorphism, then T is bijective and so T has an inverse (as a mapping) $T^{-1} : W \longrightarrow V$, it is easy to check that $T^{-1} : W \longrightarrow V$ is an isomorphism. Also composition of two isomorphisms is an isomorphism.

A semimodule over a zerosumfree semiring may have many bases. Now we show that the Definition 3.1 of inner product is independent of the choice of basis.

For this we first prove the following lemma.

Lemma 3.4. *Let V be a zerosumfree semimodule with a basis $\{v_1, \cdots, v_n\}$. Then there exists a bijective isometry $\phi : V \to \mathcal{L}_n(A)$.*
Proof. Let $\phi : V \to \mathcal{L}_n(A)$ be defined by: for $v = \sum_{i=1}^{n} a_iv_i$,

$$\phi(v) = (a_1, \cdots, a_n).$$

It is clear that ϕ is linear. Now,

$$\begin{aligned}\langle\phi(u), \phi(v)\rangle &= \langle(a_1, \cdots, a_n), (b_1, \cdots, b_n)\rangle\\ &= \sum a_ib_i = \langle u, v\rangle.\end{aligned}$$

So ϕ is isometry. That it is bijective follows from the definition of basis.

Theorem 3.5. *The definition of inner product in V is independent of the choice of the basis.*
Proof. Let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_m\}$ be two bases of V. Then the mapping

$\phi_1 : V \to \mathcal{L}_n(A)$ defined by ,

$$\phi_1(\sum a_i u_i) = (a_1, \cdots, a_n)$$

is an isometry.
Similarly $\phi_2 : V \to \mathcal{L}_m(A)$ defined by,

$$\phi_2(\sum a_i v_i) = (a_1, \cdots, a_m)$$

is also an isometry.
Let $\langle u, v\rangle_1$ and $\langle u, v\rangle_2$ be the inner products on V corresponding to the bases $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_m\}$ respectively.
Then $\langle u, v\rangle_1 = \langle \phi_1(u), \phi_1(v)\rangle$ and $\langle u, v\rangle_2 = \langle \phi_2(u), \phi_2(v)\rangle$ for all $u, v \in V$.
Since $\phi_2 \circ \phi_1^{-1} : \mathcal{L}_n(A) \to \mathcal{L}_m(A)$ is an isometry, we have

$$\begin{aligned}
&\langle u, v\rangle_1 = \langle \phi_1(u), \phi_1(v)\rangle \\
&= \langle (\phi_2 \circ \phi_1^{-1})(\phi_1(u)), (\phi_2 \circ \phi_1^{-1})(\phi_1(v))\rangle \\
&= \langle \phi_2(u), \phi_2(v)\rangle = \langle u, v\rangle_2.
\end{aligned}$$

Hence the result follows.
Orthogonality and orthonormality of vectors in a zerosumfree semimodule V are defined as in $\mathcal{L}_n(A)$ in Section 2.

Theorem 3.6. *If $T : V \longrightarrow W$ is an isometry, then T maps every orthonormal basis of V onto an orthonormal basis of $T(V)$.*
Proof. Suppose $\{v_1, \cdots, v_n\}$ is an orthonormal basis of V and $w \in T(V)$. Then there is $v \in V$ such that $w = T(v)$. Now $v = \sum_{i=1}^n a_i v_i$ for some $a_i \in A$,$i = 1, \cdots, n$ implies that $w = \sum_{i=1}^n a_i T(v_i)$ and hence every element of $T(V)$ is expressed as a linear combination of $T(v_1), T(v_2), \cdots, T(v_n)$ on A.

Now to show that such linear combinations are unique, consider $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n \in A$ such that $\sum_{i=1}^n a_i T(v_i) = \sum_{i=1}^n b_i T(v_i)$. This implies that $T(\sum_{i=1}^n a_i v_i) = T(\sum_{i=1}^n b_i v_i)$ and so $\sum_{i=1}^n a_i v_i = \sum_{i=1}^n b_i v_i$, since T is one-to-one. Again since $\{v_1, v_2, \cdots, v_n\}$ is a basis for V, we have $a_i = b_i$ for all $i = 1, 2, \cdots n$.
Indent also $\langle v_i, v_j\rangle = \delta_{ij}$, implies that $\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij}$, for all $i, j = 1, 2, \cdots, n$. Thus T carries an orthonormal basis of V onto an orthonormal basis of $T(V)$.

Corollary 3.7. *If $T : V \longrightarrow W$ is an isomorphism, then T maps every orthonormal basis of V onto an orthonormal basis of W.*

Corollary 3.8. *If there is an isomorphism $T : \mathcal{L}_m(A) \longrightarrow \mathcal{L}_n(A)$ then $m = n$.*

Proof. Consider the standard basis $\{e_1, e_2, \cdots, e_m\}$ of $\mathcal{L}_m(A)$, since $T : \mathcal{L}_m(A) \longrightarrow \mathcal{L}_n(A)$ is an isomorphism, $\{Te_1, Te_2, \cdots, Te_m\}$ is also an orthonormal basis of $\mathcal{L}_n(A)$. Again every orthonormal basis of $\mathcal{L}_n(A)$ has exactly n vectors, by Theorem 2.8. Hence $m = n$.
Now we prove our main theorem of this section.

Theorem 3.9. *Every basis of a zerosumfree semimodule have the same number of vectors.*
Proof. Let V be a semimodule over a zerosumfree semiring A. Consider two bases $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ of V. Then $\phi_2 \circ \phi_1^{-1} : \mathcal{L}_n(A) \longrightarrow \mathcal{L}_m(A)$ is an isomorphism where $\phi_1 : V \longrightarrow \mathcal{L}_n(A)$ and $\phi_2 : V \longrightarrow \mathcal{L}_m(A)$ are as in Lemma 3.5. Hence, by the Corollary 3.8, $m = n$.

Acknowledgement

The author is grateful to Dr. A. K. Bhuniya for giving the problem and helping to prepare this manuscript. Also he acknowledges the support of DST-INSPIRE Fellowship grant, India.

References

[1] Z. Q. Cao, K. H. Kim, F. W. Roush, Incline Algebra and applications, *John Wiley*, New York, 1984.

[2] S. Ghosh, Matrices over semirings, *Inform. Sci.*, **90**(1996), 221-230.

[3] J. S. Golan, Semirings and their Applications, *Kluwer Academic Publishers*, 1999.

[4] D. Gregory, N.J. Pullman and S. Kirkland, On the dimension of the algebra generated by a boolean matrix, *Linear and Multilinear Algebra*, **38(1-2)** (1994), 131-144.

[5] S. Gudder and F. Latremoliere, Boolean inner-product spaces and Boolean matrices, *Linear Algebra and its applications*, **431**(2009), 274-296.

[6] S. C. Han and H. X. Li, Invertible incline matrices and Cramer's rule over inclines, *Linear Algebra Appl.* **389**(2004), 121-138.

[7] U. Hebisch and H. J. Weinert, Semirings: Algebraic Theory and Applications in Computer Science, *World Scientific, Singapore*, 1998.

[8] E. P. Klement, R. Mesiar and E. Pap, Triangular Norms, *Springer, Netherlands*, 2000.

[9] M. Minoux, Bideterminants, arborescences and extension of the matrix-tree theorem to semirings, *Discrete Math.* **171**(1997), 191-200.

[10] M. Minoux, A generalization of the all minors matrix tree theorem to semirings, *Discrete Math.* **199**(1999), 139-150.

[11] W. Mora, A. Wasanawichit and Y. Kemprasit, Invertible Matrices over Idempotent Semirings, *Chamchuri J. Math.* **1(2)**(2009), 55-61.

[12] P. L. Poplin and R. E. Hartwig, Determinantal identities over commutative semirings, *Linear Algebra Appl.* **378**(2004), 99-132.

[13] C. Reutenauer and H. Straubing, Inversion of matrices over a commutative semiring, *J. Algebra*, **88**(1984), 350-360.

[14] Qian-yu Shu and Xue-ping Wang, Bases in semilinear spaces over zerosumfree semirings, *Linear algebra and its Applications* **435**(2011), 2681-2692.

[15] Y. Tan, On invertible matrices over antirings, *Linear Algebra and its Applications*, **423**(2007), 428-444.

[16] Yi-Jia Tan, Bases in semimodules over commutative semirings, *Linear Algebra and its Applications*, **443**(2014), 139-152

[17] S. Zhao and Xue-ping Wang, Bases in semilinear spaces over join-semirings, *Fuzzy Sets and Systems*, **182**(2011), 93-100.